UDC 62-50

ON THE CONTROL OF A HYPERBOLIC SYSTEM UNDER CONDITIONS OF UNCERTAINTY

PMM Vol. 42, No. 4, 1978, pp. 606-612

S. P. OKHEZIN
(Sverdlovsk)

(Received December 9, 1977)

A problem on the feedback control of a hyperbolic system under conditions of uncertainty or conflict is analyzed. The problem is interpreted as a position differential game [1-3] in a suitable functional space. The controls enter into the boundary conditions and the mechanism for developing these controls is described by an ordinary differential equation. The constructions are based on a approach to position control problems for distributed-parameter systems developed in [4-8]. As in the case of ordinary differential equations [1-3] the class of strategies solving the problems being examined is indicated.

1. Let Ω be a bounded connected open set in the Euclidean space R_n and Γ, the boundary of Ω, be an (n-1)-dimensional manifold. We consider the conflict-controlled system

$$\frac{\partial^2 y(t,x)}{\partial t^2} = Ay(t,x) + g(t,x) \text{ in } Q = (t_0, \vartheta) \times \Omega \tag{1.1}$$

$$y(t_0, x) = y_0(x), \quad \frac{\partial y(t_0, x)}{\partial t} = y_1(x) \text{ in } \Omega$$

$$(y_0 \in H^1(\Omega), \; y_1 \in L_2(\Omega))$$

$$\frac{\partial y(t,x)}{\partial \nu} + \sigma(x) y(t,x) = \alpha(x) w(t) \text{ in } \Sigma = (t_0, \vartheta) \times \Gamma$$

$$(\alpha(x) = \{\alpha_i(x)\}, \; \alpha_i(\cdot) \in L_2(\Gamma), \; t = 1, \ldots, m)$$

$$\frac{\partial w}{\partial t} = f(t, w) + B(t) u - C(t) v; \; w(t_0) = w_0 \tag{1.2}$$

$$Ay = \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial y}{\partial x_j}\right) + a(x) y$$

Here $a_{ij}(x) = a_{ji}(x)$ are functions continuously differentiable on $\bar{\Omega}$; a constant $\nu > 0$ exists for which the inequality

$$\nu(\xi_1^2 + \ldots + \xi_n^2) \leqslant \sum_{ij=1}^{n} a_{ij}(x) \xi_i \xi_j$$

where $a(x)$ is a function continuous on $\overline{\Omega}$, for any $x \in \Omega$ and $\xi_i \in R_1$, $i = 1, \ldots, n$. $H^1(\Omega)$ is a first-order Sobolev space on set Ω [9]. $L_2(\Omega)$ is the space (of classes) of functions (Lebesgue-) square-summable on Ω. We assume: domain Ω has a boundary Γ for which the elements of space $H^1(\Omega)$ have traces on Γ from $L_2(\Gamma)$ and the formula for integration by parts and the theorem on the compactness of the imbedding of $H^1(\Omega)$ into $L_2(\Omega)$ and into $L_2(\Gamma)$ are true for them (for example, see [9-11]); $g \in L_2(Q)$ is a specified perturbation; w is an m-dimensional phase vector of system (1.2); $\sigma(\cdot)$ is a measurable function bounded on Γ. For the sake of unessential simplifications we set (see [10-12]) $\sigma \geqslant 0$, $a \leqslant 0$, $\sigma^2 + a^2 \not\equiv 0$ on Γ. The function in the right-hand side of (1.2) is assumed to be continuous in all arguments and to satisfy a Lipschitz condition in w in each bounded domain of space R_m and the condition of uniform continuability of solutions $w(t)$ for $t \geqslant t_0$ under every choice

$$u(t) \in P(t), \quad v(t) \in Q(t) \tag{1.3}$$

where $P(t)$ and $Q(t)$ are convex compacta in Euclidean spaces R_{l_1} and R_{l_2}, respectively, measurable and equibounded with respect to $t \in [t_0, \vartheta]$; $B(t)$ and $C(t)$ are continuous matrices of appropriate dimensionalities. A closed set M is specified in space $R_1 \times H^1(\Omega) \times L_2(\Omega)$. We are required to construct a method for choosing a feedback control u (feedback control v), producing realizations $u[t]$ ($v[t]$) that are Lebesgue-measurable on $[t_0, \vartheta]$ and satisfying (1.3), for which the condition $\{t, y(t, \cdot), \partial y(t, \cdot)/\partial t\} \in M$ is fulfilled for some $t \in [t_0, \vartheta]$ (the condition $\{t, y(t, \cdot), \partial y(t, \cdot)/\partial t\} \notin M$ is fulfilled for all $t \in [t_0, \vartheta]$) for any law of formation of the measurable realization $v[t] \in Q(t)$ ($u[t] \in P(t)$).

N o t e 1.1. The smoothness of the generalized solutions of system (1.1) (see [9-12]) essentially depends upon the smoothness of the boundary conditions in Σ. Generalized solutions from the so-called "energy" classes (see [11]) are of practical interest. The introduction of an ordinary differential equation into system (1.1) is one of the possible variants for obtaining the necessary smoothness of the generalized solutions. From the mechanical point of view the problem can be interpreted as a problem of obtaining the optimal perturbed state of an oscillating body Ω under conditions of undeterminate interference by means of a mechanical force $\alpha(x)\, w(t)$ distributed on the body's boundary Γ. In this connection it is considered that the law of variation of force w is described by an ordinary differential equation.

Let us pose the problem more precisely. Let $\{P; q\}$ be a collection of all functions $u(t) \in P(t)$ measurable on a set q. A vector $r = \{w, y_1, y_2\}$, where $w \in R_m$, $y_1 \in H^1(\Omega)$ and $y_2 \in L_2(\Omega)$, is called a state of system (1.1)-(1.2). The pairs $\{t, r\}$, $t \in [t_0, \vartheta]$, are called positions. A rule associating a nonempty set

$$U(t_1, t_2, r) \subset \{P; [t_1, t_2)\} \ (V(t_1, t_2,) \subset \{Q; [t_1, t_2)\})$$

with each triple $\{t_1, t_2, r\}$, where $t_0 \leqslant t_1 < t_2 \leqslant \vartheta$ and $r = \{w, y_1, y_2\}$, is called the first (second) player's strategy U (V).

We consider the system adjoint to (1.1). When $\varphi(t, x)$ ranges space $L_2(Q)$ (see [9, 11]) the solution of the adjoint system (see [9, 11, 12]) ranges a set X to which we allot a topology introduced by the mapping $\varphi \to z(\cdot, \cdot; \varphi)$, where $z(\cdot, \cdot; \varphi)$ is a solution of the adjoint system. Let Δ be a partitioning of interval $[t_0, \vartheta]$ by points τ_i, $i = 1, \ldots, m(\Delta)$ $(\tau_{i+1} > \tau_i,\ \tau_1 = t_0,\ \tau_{m(\Delta)} = \vartheta)$ and let $\delta(\Delta) = \max_i(\tau_{i+1} - \tau_i)$. The pair $\{y(t, \cdot)_\Delta,\ \partial y(t, \cdot)_\Delta / \partial t\} = \{y(t, x;\ t_0, w_0, y_0, y_1, U)_\Delta,\ y_t'(t, x;\ t_0, w_0, y_0, y_1, U)_\Delta\}$, $x \in \Omega$ is called a motion of system (1.1) from position $\{t_0, w_0, y_0, y_1\}$, corresponding to strategy U and partitioning Δ, if

$$y(t, \cdot)_\Delta \in C^0([t_0, \vartheta]; H^1(\Omega)) \cap C^1([t_0, \vartheta]; L_2(\Omega)) \tag{1.4}$$

$$\int_{t_0}^{\vartheta}\int_{\Omega} y(t, x)_\Delta \left(\frac{\partial^2 \varphi(t, x)}{\partial t^2} - A\varphi(t, x)\right) dx\,dt = \int_{t_0}^{\vartheta}\int_{\Omega} g(t, x)\varphi(t, x)\,dx\,dt - \tag{1.5}$$
$$\int_{\Omega} y_0(x)\frac{\partial \varphi(t_0, x)}{\partial t}\,dx + \int_{\Omega} y_1(x)\varphi(t_0, x)\,dx +$$
$$\int_{t_0}^{\vartheta}\int_{\Gamma} \alpha(x) w(t)\varphi(t, x)\,d\Gamma\,dt, \quad \forall \varphi \in X$$

Here $w(t)$ is a solution of the integral equation

$$w(t) = w_0 + \int_{t_0}^{t} \{f(\tau, w(\tau)) + B(\tau)u(\tau) - C(\tau)v(\tau)\}\,d\tau \tag{1.6}$$

where on $[\tau_i, \tau_{i+1}]$ the controls $u(\cdot) \in U(\tau_i, \tau_{i+1}, w(\tau_i), y(\tau_i, \cdot)_\Delta, y_t'(\tau_i, \cdot)_\Delta)$ and $v(\cdot) \in \{Q;\ [\tau_i, \tau_{i+1})\}$, $i = 1, \ldots, m(\Delta) - 1$; $y_t'(t, \cdot)_\Delta$ is the derivative with respect to t of $y(t, \cdot)_\Delta$ as an element of space $C^1([t_0, \vartheta];\ L_2(\Omega))$. The motion $\{y(t, x;\ t_0, w_0, y_0, y_1, V)_\Delta, y_t'(t, x;\ t_0, w_0, y_0, y_1, V)_\Delta\}$ is defined similarly. Here $C^k([t_0, \vartheta];\ X)$ is the space of functions k times continuously differentiable on $[t_0, \vartheta]$, with values in space X. The set of motions introduced is not empty.

N o t e 1.2. The existence of function $y(t, x)_\Delta \in L_2(Q)$, satisfying the integral indentity (1.5), follows from the results in [9, 11]. To prove that $y(\cdot, \cdot)_\Delta \in C^0([t_0, \vartheta];\ H^1(\Omega)) \cap C^1([t_0, \vartheta];\ L_2(\Omega))$ we use the expansion of the solution of (1.5) into a series in the eigenfunctions of the spectral problem

$$A\omega = -\lambda\omega, \quad \partial\omega / \partial\nu_A + \sigma\omega|_\Gamma = 0$$

and the estimate from [13].

This problem has a solution from $H^1(\Omega)$ for a denumerable number of values of λ (see [10, 11]). We note that under the assumptions made, all $\lambda_j > 0$.

In space $H^1(\Omega)$ we introduce the norm

$$\| h \|_1^2 = \int_\Omega \sum_{ij=1}^n a_{ij}(x) \frac{\partial h}{\partial x_j} \frac{\partial h}{\partial x_i}\, dx - \int_\Omega a(x) h^2\, dx + \int_\Gamma \sigma h^2\, d\Gamma$$

equivalent to the norm of space $H^1(\Omega)$ in [10, 11]. We formalize the initial problems in the following manner. Let M^ε be a closed ε-neighborhood of set M in the space $R_1 \times H^1(\Omega) \times L_2(\Omega)$.

Problem 1 (the encounter problem). Construct a strategy U with the property: for any $\varepsilon > 0$ we can find a positive number δ_0 for which the inclusion $\{t, y(t, \cdot)_\Delta, y_t'(t, \cdot)_\Delta\} \in M^\varepsilon$ is fulfilled at some $t \in [t_0, \vartheta]$ for all motions $\{y(t, x; t_0, w_0, y_0, y_1, U)_\Delta, y_t'(t, x; t_0, w_0, y_0, y_1, U)_\Delta\}$ if only $\delta(\Delta) \leqslant \delta_0$.

Problem 2 (the evasion problem). Construct a strategy V with the property: for any $\varepsilon > 0$ we can find a positive number δ_0 for which the inclusion $\{t, y(t, \cdot)_\Delta, y_t'(t, \cdot)_\Delta\} \notin M^\varepsilon$ is fulfilled at all $t \in [t_0, \vartheta]$ for all motions $\{y(t, x; t_0, w_0, y_0, y_1, V)_\Delta, y_t' \subset (t, x; t_0, w_0, y_0, y_1 V)_\Delta\}$ if only $\delta(\Delta) \leqslant \delta_0$.

2. The following basic theorem is valid.

Theorem 2.1 (on the alternative in an encounter-evasion game). One and only one of the following statements is valid for any initial position $\{t_0, w_0, y_0, y_1\}$, $w_0 \in R_m$ and $\{y_0, y_1\} \in H^1(\Omega) \times L_2(\Omega)$, any instant $\vartheta \geqslant t_0$ and any set $M \subset R_1 \times H^1(\Omega) \times L_2(\Omega)$

1) a first player's strategy U exists, solving the encounter problem for the data mentioned;

2) a second player's strategy V exists, solving the evasion problem for the data mentioned.

Let us consider the basic constructions that can be used to prove Theorem 2.1. By H we denote the space

$$H = R_m \times H^1(\Omega) \times L_2(\Omega)$$

with the norm

$$\| \{w, y_0, y_1\} \|_H = (\| w \|_{R_m}^2 + \| y_0 \|_{H^1(\Omega)}^2 + \| y_1 \|_{L_2(\Omega)}^2)^{1/2}$$

We define set M° as

$$M^\circ = \{\{t, w, y_1, y_2\} \in [t_0, \vartheta] \times H \mid \{t, y_1, y_2\} \in M\}$$

We introduce the concept of stable sets by analogy with [1,4-8]. Let K_1 and K_2 be certain collections of pairs $\{t, h\}$, $t \in [t_0, \vartheta]$ and $h \in H$. We say that set K_1 is u-stable relative to set K_2 if for every choice of the pair $\{t_*, h_*\}$ $\in K_1$, $h_* = \{w_*, y_*^1, y_*^2\}$, the instant $t^* > t_*$ and the control $v(\cdot) \in \{Q; [t_*, t^*)\}$ we can find at least one control $u(\cdot) \in \{P; [t_*, t^*)\}$ under which the inclusion

$$\{t^*, r[t^*]\} \in K_1, \quad \{\tau, r[\tau]\} \in K_2 \tag{2.1}$$

is valid at some $\tau \in [t_*, t^*]$ for the function $r[t] = \{w[t], y[t, \cdot], y_t' \cdot [t, \cdot]\}$. Here $w[t]$ is a solution of Eq. (1.4) with initial condition w_* and controls $u(\cdot)$ and $v(\cdot)$; $\{y[t, \cdot], y_t'[t, \cdot]\}$ is the corresponding motion of system (1.1) from the position $\{t_*, w_*, y_*^1, y_*^2\}$ on the interval $[t_*, t^*]$. The concept of a v-stable bridge is introduced similarly.

Let K be an arbitrary set in the space of positions $\{t, r\}$ $\in [t_0, \vartheta] \times H$. We construct the first player's strategy U^e which we say is extremal to set K. By J we denote the following functional on space $H \times H$:

$$J(r_1, r_2) = \langle w_1, w_2\rangle_{R_m} + \langle y_{11}, y_{12}\rangle_{L_2(\Omega)} + \langle y_{21}, y_{22}\rangle_{H^{-1}(\Omega)}$$

$$J_0(r) = J(r, r) \quad (H^{-1}(\Omega) = (H^1(\Omega))^*; \ r_i = \{w_i, y_{1i}, y_{2i}\})$$

By $K(t)$ we denote the section of set K by the hyperplane $\tau = t$. Suppose that some triple $\{t_*, t^*, r\}$, has been selected, where $t_0 \leqslant t_* < t^* \leqslant \vartheta$ and $r = \{w, y_1, y_2\} \in H$. . If $K(t_*) \neq \phi$, then by $U^e(t_*, t^*, r)$we denote the collection of all $u^e(\cdot) \in \{P; [t_*, t^*)\}$ with the property:

1) a sequence of function $u^{(n)}(\cdot) \in \{P; [t_*, t^*)\}$converging weakly in space $L_2([t_*, t^*); R_{l_1})$ to function $u^e(\cdot)$; can be found;

2) a sequence of $r_n \in K(t_*)$, $r_n = \{w^n, z_1^n, z_2^n\}$, can be found, for which

$$\lim_{n\to\infty} J_0(r - r_n) = \inf\{J_0(r - r_\alpha) \mid r_\alpha \in K(t_*)\} \tag{2.2}$$

3) for each $n = 1, 2, \ldots$

$$\left\langle w - w^n, \int_{t_*}^{t^*} B(t) u^n(t)\, dt \right\rangle_{R_m} = \min\left\{\left\langle w - w^n, \int_{t_*}^{t^*} B(t) u(t)\, dt\right\rangle_{R_m} \mid u(\cdot) \in \{P; [t_*, t^*)\}\right\} \tag{2.3}$$

The set of $U^e(t_*, t^*, r)$ is nonempty since $\{P; [t_*, t^*)\}$ is a subject, weakly compact in itself, of space $L_2([t_*, t^*); R_{l_1})$. The second player's strategy V^e extremal to set K is defined similarly.

By V_0 we denote strategies upper-semicontinuous in variable r.

The latter means that for any quantities $t_0 \leqslant t_* < t^* \leqslant \vartheta$ and $r \in H$ the inclusion $v \in V_0(t_*, t^*, r)$ follows from the conditions $r_k \to r$ in H, $v_k \to v$

weakly in $L_2([t_*, t^*); R_{l_2})$, $v_k \in V_0(t_*, t^*, r_k)$. By a motion of system (1.1)-(1.2), corresponding to strategy U (V) and partitioning Δ we mean the triple $\{w[t]_\Delta, y[t, \cdot]_\Delta, y_t'[t, \cdot]_\Delta\}$, where the pair $\{y[t, \cdot]_\Delta, y_t'[t, \cdot]_\Delta\}$ determines the motion of system (1.1) and $w[t]_\Delta$ is determined from (1.6). By $W(M^\circ)$ we define the collection of all pairs $\{t_*, r_*\} \in [t_0, \vartheta] \times H$ possessing the property: for any strategy V_0, and any numbers $\varepsilon > 0$ and $\delta > 0$ there exists at least one motion $\{w[t; t_*, r_*, V_0]_\Delta, y[t, x; t_*, r_*, V_0]_\Delta, y_t'[t, x; t_*, r_*, V_0]_\Delta\}$, corresponding to a partitioning Δ with diameter $\delta(\Delta) \leqslant \delta$ of the interval $[t_*, \vartheta]$ for which the inclusion

$$\{t, w[t]_\Delta, y[t, \cdot]_\Delta, y_t'[t, \cdot]_\Delta\} \in M^{\circ\varepsilon}$$

is fulfilled at some instant $t \in [t_*, \vartheta]$. Here $M^{\circ\varepsilon}$ is a closed ε-neighborhood of set M° in the metric $\|\{t, r\}\| = (t^2 + \|r\|_H^2)^{1/2}$.

L e m m a 2.1. Set $W(M^\circ)$ is u-stable relative to set M°.

This lemma can be proved by the proof scheme for analogous statements in [1,4-8].

N o t e 2. 1. Since the sheaf of motions of system (1.1) under controls u and v satisfying (1.3) is a compactum in $C^\circ([t_0, \vartheta]; H^1(\Omega) \times L_2(\Omega))$, without loss of generality we can assume that set M is a compactum in $R_1 \times H^1(\Omega) \times L_2(\Omega)$.

T h e o r e m 2.2. Problem 1 on encounter has a solution if and only if the condition

$$\{w_0, y_0, y_1\} \in W(M^\circ)(t_0) \tag{2.4}$$

is fulfilled for the initial position $\{t_0, w_0, y_0, y_1\}$. Under inclusion (2.4) the solution of the problem is provided by the strategy U^e extremal to set $W(M^\circ)$.

Let us consider the main features of the proof of this theorem. Let $r^e[t]_\Delta = \{w[t]_\Delta, y[t, \cdot]_\Delta, y_t'[t, \cdot]_\Delta\}$ be the motion of system (1.1)-(1.2), corresponding to the extremal strategy U^e and the selected partitioning $\Delta(\tau_i)$ of the interval $[t_0, \vartheta]$. We introduce the following functionals

$$\varepsilon[t] = \begin{cases} \inf\{J_0(r^e[t]_\Delta - r) \mid r \in W(M^\circ)(t), W(M^\circ)(t) \neq \phi \\ \infty, \ W(M^\circ)(t) = \phi \end{cases}$$

$$\gamma[t] = \begin{cases} \inf\{J_0(r^e[t]_\Delta - r) \mid r \in M^\circ(t)\}, \ M^\circ(t) \neq \phi \\ \infty, \ M^\circ(t) = \phi \end{cases}$$

Taking into account the definition of extremal strategy U^e and the stability of set $W(M^\circ)$, we obtain the estimates

$$\begin{gathered} \varepsilon^2[\tau_{i+1}] \leqslant \varepsilon^2[\tau_i] \ (1 + K\delta(\Delta)) + o(\delta(\Delta)) \\ \gamma^2[t^{(k)}] \leqslant 2J_0 \times (r^e[\tau_i]_\Delta - r_k)(1 + N\delta(\Delta)) + o(\delta(\Delta)) + \varphi(k) \\ (K, N = \text{const} > 0; \ \varphi(k) \to 0, \ k \to \infty; \ o(\delta(\Delta)) / \delta(\Delta) \to 0, \ \delta(\Delta) \to 0) \end{gathered} \tag{2.5}$$

for $\varepsilon[\tau_{i+1}]$ and $\gamma[t^{(k)}]$. Here $t^{(k)} \in [\tau_i, \tau_{i+1}]$ satisfies the second inclusion in (2.1) and the quantities r_k are determined from (2.2). The theorem's assertion follows from estimates (2.5) and the compactness of the sheaf of motions of system (1.1) in $R_1 \times H^1(\Omega) \times L_2(\Omega)$.

Note 2.2. The strategy V yielding the solution to Problem 2 can be constructed as a strategy extremal to a certain v-stable set.

As in [1, 6-8] we can delineate a broader class of sets M (for example, sets convex, closed and bounded in $R_1 \times H^1(\Omega) \times L_2(\Omega)$ for which the set $W(M^\circ)$ admits of an effective description in the form of linear inequalities in $R_1 \times H^1(\Omega) \times L_2(\Omega)$).

The author thanks Iu. S. Osipov for the formulation of the problem and for valuable advice.

REFERENCES

1. Krasovskii, N. N. and Subbotin, A. I., Position Differential Games. Moscow "Nauka", 1974.

2. Pontriagin, L. S., On linear differential games. Dokl. Akad. Nauk SSSR, Vol. 174, No. 6, 1967.

3. Mishchenko, E. F., Problems of pursuit and contact-evasion in the theory of differential games. Izv. Akad. Nauk SSSR, Tekhn. Kibernet., No. 5, 1971.

4. O s i p o v, Iu. S., Differential games for systems with aftereffect. Dokl. Akad. Nauk SSSR, Vol. 196, No. 4, 1971.

5. O s i p o v, Ius. S., On the theory of differential games in distributed-parameter systems. Dokl. Akad. Nauk SSSR, Vol.223, No. 6, 1975.

6. V á i s b u r d, I. F. and Osipov, Iu. S., Differential games of encounter for distributed-parameters systems. PMM Vol. 39, No. 5, 1975.

7. O s i p o v, Iu. S. and Okhezin, S. P., On the theory of differential games in parabolic systems. Dokl. Akad. Nauk SSSR, Vol. 226, No. 6, 1976.

8. O s i p o v, Iu. S. and Okhezin, S. P., On the theory of position control in hyperbolic systems. Dokl. Akad. Nauk SSSR, Vol. 233, No. 4, 1977.

9. L i o n s, J.-L., Optimal Control of Systems Governed by Partial Differential Equations. New York-Berlin, Springer-Verlag, 1971.

10. L a d y z h e n s k a i a, O. A. and Ural'tseva, N. N., Linear and Quasilinear Equations of Elliptic Type. Moscow, "Nauka", 1973.

11. L a d y z h e n s k a i a, O. A., Boundary-Value Problems of Mathematical Physics. Moscow, "Nauka", 1973.

12. L a d y z h e n s k a i a, O. A., Mixed Boundary-Value Problem for a Hyperbolic Equation. Moscow, Gostekhizdat, 1953.

13. P l o t n i k o v, V. I., Power inequality and property of uncertainty of a system of eigenfunctions. Izv. Akad. Nauk SSSR, Ser. Mat., Vol. 32, No. 4, 1968.

Translated by N. H. C.